Announcements

1) Office hours as usual this week

2) Exam 2 moved back one week to March 20

Recall: The idea of a vector space:

There is a set V on which you can add and scalar multiply; there is a "zero vector" 0_V

Examples: $\mathbb{R}^n$, $M_n(\mathbb{R})$, $C(\mathbb{R})$ (continuous functions from $\mathbb{R}$ to $\mathbb{R}$)

Definition: (subspace)

A subset W of a vector space V (written "$W \subseteq V$") is a subspace of V if W is a vector space under the same operations and with the same zero vector as V.

Subspace Test

$W \subseteq V$ is a subspace of V precisely when,

for all x, y in W and scalars c,

1) 0_V is in W

2) cx is in W

3) $\underbrace{x - y \text{ is in } W}_{\text{same as } -y,\ x+y \text{ is in } W}$

Example 1: (polynomials)

Remember that if

$a_0, a_1, \ldots, a_n$ are real

numbers, we define a

polynomial by

$$\boxed{P(x) = \sum_{k=0}^{n} a_k x^k}$$

where x is a real number.

If W is the set of all polynomials and V is the vector space $C(\mathbb{R})$, then every polynomial is continuous, $W \subseteq V$.

Is W a subspace of V?

Use the subspace test.

Is 0_V in W?

Yes - the zero vector is the function $f(x)=0$ for all x in $\mathbb{R}$. This is a polynomial (of degree zero).

If p is a polynomial and c is a real number, is cp a polynomial?

Let $p(x) = \sum_{k=0}^{n} a_k x^k$.

$$cp(x) = c\left(\sum_{k=0}^{n} a_k x^k\right)$$

$$= \sum_{k=0}^{n} (c a_k) x^k$$

This is a polynomial

$$\sum_{k=0}^{n} b_k x^k \quad \text{where}$$

$$b_k = c\,a_k \, .$$

If p and q are polynomials, is $p-q$ a polynomial?

Let $p(x) = \sum_{k=0}^{n} a_k x^k$ and

$$q(x) = \sum_{k=0}^{m} b_k x^k.$$

One of either m or n is bigger than or equal to the other. Say $n \geq m$.

Add phantom coefficients to q by setting $b_k = 0$ for all k, $m < k \leq n$.

Then $p(x) - q(x)$

$$= \sum_{k=0}^{n} a_k x^k - \sum_{k=0}^{n} b_k x^k$$

$$= \sum_{k=0}^{n} (a_k - b_k) x^k$$

This is a polynomial

$$\sum_{k=0}^{n} c_k x^k$$ with

$$c_k = a_k - b_k .$$

Therefore, W is a subspace of $C(\mathbb{R})$!

Note: All polynomials of degree n are denoted by P_n.

P_n is a subspace of the vector space of polynomials P for every n.

Example 2: $(C_1(\mathbb{R}))$

$C_1(\mathbb{R})$ = all differentiable functions from $\mathbb{R}$ to $\mathbb{R}$.

Every differentiable function is continuous, so if $W = C_1(\mathbb{R})$, $W \subseteq V$ where $V = C(\mathbb{R})$.

Is W a subspace?

Is 0_V in W?

Is zero differentiable?

Yes, with derivative zero!

So 0_V is in W.

If f is differentiable and c is a real number, is cf differentiable?

Yes, since

$$(cf)'(x) = cf'(x).$$

(rules of differentiation)

If f and g are differentiable, is $f-g$ differentiable?

Yes, since

$$(f-g)'(x) = f'(x) - g'(x).$$

So $C_1(\mathbb{R})$ is a subspace of $C(\mathbb{R})$.

Let $C_n(\mathbb{R})$ denote the set of functions from $\mathbb{R}$ to $\mathbb{R}$ that are differentiable n times.

Then $C_n(\mathbb{R})$ is a subspace of $C(\mathbb{R})$ and

$$
\begin{aligned}
C(\mathbb{R}) \supseteq C_1(\mathbb{R}) &\supseteq C_2(\mathbb{R}) \\
&\supseteq C_3(\mathbb{R}) \\
&\;\vdots \\
&\supseteq C_n(\mathbb{R})
\end{aligned}
$$

Note $C_\infty(\mathbb{R}) =$

all infinitely differentiable

functions.

$C_\infty(\mathbb{R})$ is a subspace

of $C_n(\mathbb{R})$ for any

value of n.

Example 3: (subspace of $\mathbb{R}^3$)

Define

W as the subset of $\mathbb{R}^3$

consisting of all vectors

$v = (x, y, z)$ such that

$$8x - 113y - 56z = 0.$$

Is W a subspace of $\mathbb{R}^3$?

Is $(0,0,0)$ in W?

Yes, since

$$8 \cdot 0 - 113 \cdot 0 - 56 \cdot 0 = 0 \checkmark$$

If (x,y,z) is in W and c is a scalar, is $c(x,y,z)$ in W?

Yes, since $c(x,y,z) = (cx, cy, cz)$

and

$$8(cx) - 113(cy) - 56(cz)$$

$$= c(8x - 113y - 56z)$$

$$= c \cdot 0 = 0$$ since (x,y,z) is in W.

If (x_0, y_0, z_0) and (x_1, y_1, z_1) are in W, is $(x_0+x_1, y_0+y_1, z_0+z_1)$ in W?

Yes - check!